

Normalized Classes of Generalized Burgers Equations

Oleksandr A. Poheketa

Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska Str., 01601 Kyiv, Ukraine

Email: poheketa@yandex.ua

A hierarchy of normalized classes of generalized Burgers equations is studied. The equivalence groupoids of these classes are computed. The equivalence groupoids of classes of linearizable generalized Burgers equations are related to those of the associated linear counterparts using the Hopf–Cole transformation.

1 Introduction

Consider some generalizations of the prominent Burgers equation

$$u_t + uu_x + u_{xx} = 0, \quad (1)$$

which has been widely used as a one-dimensional turbulence model [1]. A review of its properties can be found in [24, Chapter 4]. The Burgers equation can be generalized in various ways. The purpose of this paper is to study a hierarchy of classes of generalized Burgers equations. One may suppose that there are few normalized ones among them. We show that the majority of naturally arising classes are normalized, which considerably simplifies the solution of the group classification problems for these classes. Namely, the problems reduce to subgroup analysis of the corresponding equivalence groups.

A class of differential equations is said to be *normalized* if its equivalence groupoid is generated by its equivalence group [15, 13, 12, 17]. The *equivalence groupoid* of a class of differential equations is the set of admissible transformations in this class with the natural groupoid structure, where the composition of mappings is the groupoid operation [14, p. 7]. An *admissible transformation* is a triple of an initial equation, a target equation and a mapping between them.

The notion of normalized classes is quite natural and useful for applications. For a normalized class of differential equations 1) its complete group classification coincides with its preliminary group classification and 2) there are no additional equivalence transformations between cases of the classification list. This notion can be weakened. For example, weakly normalized classes maintain the first of the aforementioned features but may lose the second, and for semi-normalized classes the situation is opposite (see [14, 17] for precise definitions).

Hierarchies of normalized subclasses arise in the course of solving group classification problems. Observe that a single differential equation forms a normalized class. Any set of all possible equations with a prescribed number of independent variables and a fixed equation order is a normalized class likewise.

In order to prove the normalization property of a class of differential equations we compare its equivalence group with its equivalence groupoid. Practically, a class is normalized if there are no classifying conditions among the determining equations for admissible transformations. A classifying condition is, roughly speaking, a determining equation that simultaneously involves arbitrary elements of the class and parameters of admissible transformations and leads to a furcation while solving the determining equations.

Section 2 is devoted to a normalized superclass, which contains all other classes under consideration. In Section 3 we consider the relation between equivalence groupoids of classes of linear (1+1)-dimensional evolution equations and those of the associated classes of equations

linearized by the Hopf–Cole transformation $u = 2v_x/v$. In Section 4 we consider classes of generalized Burgers equations with variable diffusion coefficients. One of these classes is not normalized but it can be partitioned into two normalized subclasses. Section 5 treats the classical Burgers equation as a normalized class.

2 Normalized superclass

It is well known that the t -component of every point (or even contact) transformation between any two fixed (1+1)-dimensional evolution equations depends only on t [9, 10]. Moreover, as proved in [6, Lemma 2], any point transformation between two equations from the class

$$u_t = F(t, x, u)u_{xx} + G(t, x, u, u_x) \quad (2)$$

has the form $\tilde{t} = T(t)$, $\tilde{x} = X(t, x)$, and $\tilde{u} = U(t, x, u)$ with $T_t X_x U_u \neq 0$. The coefficients F and G are arbitrary smooth functions of their arguments with $F \neq 0$.

This class is normalized in the usual sense [6], and any contact transformation between equations from it is generated by a point transformation [19]. However, class (2) is too wide for the generalized Burgers equations. For our purpose it is more convenient to consider its subclass,

$$u_t + F(t, x, u)u_{xx} + H^1(t, x, u)u_x + H^0(t, x, u) = 0, \quad (3)$$

where the coefficients F , H^1 , and H^0 are arbitrary smooth functions of their arguments with $F \neq 0$. This class is considered as the initial superclass for the present paper. As it contains all subclasses to be studied, any transformation between two fixed equations from each specified subclass obeys the restrictions marked for class (2).

In order to find the general form of admissible transformations for class (3), we write an equation of this class in tilded variables, $\tilde{u}_{\tilde{t}} + \tilde{F}\tilde{u}_{\tilde{x}\tilde{x}} + \tilde{H}^1\tilde{u}_{\tilde{x}} + \tilde{H}^0 = 0$, and replace $\tilde{u}_{\tilde{t}}$, $\tilde{u}_{\tilde{x}}$, and $\tilde{u}_{\tilde{x}\tilde{x}}$ with their expressions in terms of untilded variables. After restricting the result to the manifold defined by the initial equation using the substitution $u_t = -Fu_{xx} - H^1u_x - H^0$, we split it with respect to u_{xx} and u_x and obtain the determining equations for admissible transformations. They imply

$$\begin{aligned} \tilde{t} &= T(t), & \tilde{x} &= X(t, x), & \tilde{u} &= U(t, x, u) = U^1(t, x)u + U^0(t, x), \\ \tilde{F} &= \frac{X_x^2}{T_t}F, & \tilde{H}^1 &= \frac{1}{T_t} \left(X_x H^1 + X_{xx} F - 2X_x \frac{U_x^1}{U^1} F + X_t \right), \\ \tilde{H}^0 &= U^1 H^0 + \frac{2U_x U_x^1}{T_t U^1} F - \frac{1}{T_t} (U_t + F U_{xx} + H^1 U_x), \end{aligned} \quad (4)$$

where $T = T(t)$, $X = X(t, x)$, $U^1 = U^1(t, x)$, and $U^0 = U^0(t, x)$ are arbitrary smooth functions of their arguments with $T_t X_x U^1 \neq 0$. Note that we obtain no additional equations (classifying conditions) on the arbitrary elements. This means that all admissible transformations in this class are generated by the transformations from the corresponding equivalence group, so class (3) is normalized.

To derive admissible transformations of any subclass of (3) it is sufficient to specify the arbitrary elements F , H^1 , H^0 , \tilde{F} , \tilde{H}^1 , and \tilde{H}^0 .

3 Linearizable generalized Burgers equations

We relate the equivalence groupoids of the class of second-order linear evolution equations and the class of linearizable generalized Burgers equations. These equations have the forms

$$v_t + a(t, x)v_{xx} + b(t, x)v_x + c(t, x)v = 0, \quad (5)$$

$$u_t + au_{xx} + (au + a_x + b)u_x + \frac{1}{2}a_x u^2 + b_x u + f = 0, \quad (6)$$

respectively. Here a, b, c are smooth functions of (t, x) with $a \neq 0$, and $f = 2c_x$. Class (6) is the widest class of differential equations that can be linearized to linear equations of form (5) by the Hopf–Cole transformation $u = 2v_x/v$. This linearization was implicitly presented in [4, p. 102, Exercise 3]. Class (6) is a subclass of (3), where the arbitrary elements are specified as $F = a$, $H^1 = au + a_x + b$, and $H^0 = \frac{1}{2}a_x u^2 + b_x u + f$. Substituting these and the corresponding tilded expressions into (4) and splitting the result with respect to u , we derive the general form of admissible transformations between two equations from class (6),

$$\begin{aligned} \tilde{t} &= T(t), & \tilde{x} &= X(t, x), & \tilde{u} &= \frac{1}{X_x}u + U^0(t, x), \\ \tilde{a} &= \frac{X_x^2}{T_t}a, & \tilde{b} &= \frac{1}{T_t}(X_x b + X_{xx}a - X_x^2 U^0 a + X_t), \\ \tilde{f} &= \frac{f}{T_t} - \frac{(X_x U^0 b)_x}{T_t} + \frac{(X_x U^0)^2 - 2(X_x U^0)_x a_x}{2T_t} + \\ &\quad + \frac{X_x U^0 (X_x U^0)_x - (X_x U^0)_{xx} a - (X_x U^0)_t}{T_t}, \end{aligned} \quad (7)$$

where $T = T(t)$, $X = X(t, x)$, and $U^0 = U^0(t, x)$ are arbitrary smooth functions of their arguments with $T_t X_x \neq 0$. There are no classifying conditions, so, transformations (7) form the (usual) equivalence group, and class (6) is normalized (in the usual sense).

Arbitrary elements of class (6) can be gauged to simple fixed values by equivalence transformations. At the first step we set $a = 1$ using the transformation

$$\tilde{t} = t \operatorname{sign} a(t, x), \quad \tilde{x} = \int \frac{dx}{\sqrt{|a(t, x)|}}, \quad \tilde{u} = u.$$

Thereby we obtain the class of equations of the general form

$$u_t + u_{xx} + (u + b)u_x + b_x u + f = 0, \quad (8)$$

where $b = b(t, x)$ and $f = f(t, x)$ are arbitrary smooth functions. The linear counterpart of (8) is $v_t + v_{xx} + bv_x + (\frac{1}{2} \int f dx) v = 0$.

The equivalence group of class (8) can be calculated directly or by means of the substitution $a = \tilde{a} = 1$ into (7). It consists of the transformations

$$\begin{aligned} \tilde{t} &= T(t), & \tilde{x} &= \varepsilon \left(\sqrt{T_t} x + X^0(t) \right), & \tilde{u} &= \varepsilon \left(\frac{1}{\sqrt{T_t}} u + U^0(t, x) \right), \\ \tilde{b} &= \varepsilon \left(\frac{b}{\sqrt{T_t}} + \frac{T_{tt}}{T_t^{3/2}} x + \frac{X_t^0}{\sqrt{T_t}} - U^0 \right), \\ \tilde{f} &= \varepsilon \left(\frac{f}{T_t^{3/2}} - \frac{(U^0 b)_x}{T_t} + \frac{U^0 U_x^0}{\sqrt{T_t}} - \frac{U_t^0}{T_t} - \frac{U_{xx}^0}{T_t} - \frac{T_{tt} U^0}{2T_t^2} \right), \end{aligned} \quad (9)$$

where $T = T(t)$, $X^0 = X^0(t)$, and $U^0 = U^0(t, x)$ are arbitrary smooth functions with $T_t > 0$, and the constant ε takes the values 1 and -1 . Class (8) is normalized.

As the next step we set the arbitrary element b to zero by means of the transformation

$$\tilde{t} = t, \quad \tilde{x} = x, \quad \tilde{u} = u + b, \quad \tilde{f} = f - b_t - bb_x - b_{xx},$$

which leads to the simplest reduced form for linearizable generalized Burgers equations containing the single arbitrary smooth function $f = f(t, x)$,

$$u_t + u_{xx} + uu_x + f = 0. \quad (10)$$

Substituting $b = \tilde{b} = 0$ into (9) we derive the general form of admissible transformations between two equations of form (10),

$$\begin{aligned} \tilde{t} &= T(t), & \tilde{x} &= \varepsilon \left(\sqrt{T_t} x + X^0(t) \right), & \tilde{u} &= \varepsilon \left(\frac{1}{\sqrt{T_t}} u + \frac{T_{tt}}{2T_t^{3/2}} x + \frac{X_t^0}{T_t} \right), \\ \tilde{f} &= \varepsilon \left(\frac{1}{T_t^{3/2}} f + \frac{3T_{tt}^2 - 2T_t T_{ttt}}{4T_t^{7/2}} x + \frac{X_t^0 T_{tt} - X_{tt}^0 T_t}{T_t^3} \right), \end{aligned}$$

where $T(t)$ is a monotonically increasing smooth function, $X^0(t)$ is an arbitrary smooth function, and $\varepsilon = \pm 1$. Class (10) is normalized. Its linear counterpart consists of equations of the form $v_t + v_{xx} + \left(\frac{1}{2} \int f dx\right) v = 0$.

Every equation from class (6) (resp. (8) or (10)) is connected with its linear counterpart via the Hopf–Cole transformation, as well as the admissible transformations in any of these classes are connected with transformations in the corresponding linear classes.

Consider now the equivalence groupoid of the class of linear equations (5). It is determined by the transformations [18]

$$\begin{aligned} \tilde{t} &= T(t), & \tilde{x} &= X(t, x), & \tilde{v} &= V^1(t, x)v + V^0(t, x), \\ \tilde{a} &= \frac{X_x^2}{T_t} a, & \tilde{b} &= \frac{1}{T_t} \left(X_x b + X_{xx} a - \frac{2X_x V_x^1}{V^1} a + X_t \right), \\ \tilde{c} &= \frac{1}{T_t} \left(c - \frac{V_x^1}{V^1} b + \frac{2(V_x^1)^2 - V^1 V_{xx}^1}{(V^1)^2} a - \frac{V_t^1}{V^1} \right), \end{aligned} \quad (11)$$

where $T = T(t)$, $X = X(t, x)$, $V^1 = V^1(t, x)$, and $V^0 = V^0(t, x)$ are arbitrary smooth functions of their arguments satisfying $T_t X_x V^1 \neq 0$ and the classifying condition

$$\left(\frac{V^0}{V^1} \right)_t + a \left(\frac{V^0}{V^1} \right)_{xx} + b \left(\frac{V^0}{V^1} \right)_x + c \frac{V^0}{V^1} = 0.$$

This means that V^0/V^1 is a solution of the initial equation (5). The equivalence group G^\sim of class (5) consists of the transformations of form (11) with $V^0 = 0$. Class (5) is not normalized but semi-normalized because every transformation of form (11) is a composition of the Lie symmetry transformation $\bar{v} = v + V^0/V^1$ of the initial equation and an element of G^\sim , namely the transformation (11) with $V^0 = 0$.

A correspondence between the equivalence groupoids (resp. groups) of classes (5) and (6) can be established using the Hopf–Cole transformation. Indeed,

$$\tilde{u} = 2 \frac{\tilde{v}_{\tilde{x}}}{\tilde{v}} = \frac{2}{X_x} \frac{V^1 v_x + V_x^1 v + V_x^0}{V^1 v + V^0} = \frac{1}{X_x} \frac{(V^1 u + 2V_x^1) v + 2V_x^0}{V^1 v + V^0},$$

which writes in terms of (t, x, u) only if $V^0 = 0$. The transformation component for u in this case is

$$\tilde{u} = \frac{1}{X_x} u + \frac{2V_x^1}{X_x V^1}, \quad \text{i.e.} \quad U^0 = \frac{2V_x^1}{X_x V^1}.$$

The constraint on V^0 is related to the general form of transformations from the equivalence group of class (5). The admissible transformations with $V^0 \neq 0$ in class (5) have no counterparts in the equivalence groupoid of class (6).

Roughly speaking, the semi-normalization of class (5) of linear equations induces the normalization of class (6) of linearizable equations.

4 Generalized Burgers equations with arbitrary diffusion coefficient

Now we set $F = f(t, x)$, $H^1 = u$, and $H^0 = 0$ in (3). This leads to the class of generalized Burgers equations with an arbitrary nonvanishing smooth coefficient $f = f(t, x)$ of u_{xx} ,

$$u_t + uu_x + f(t, x)u_{xx} = 0. \quad (12)$$

Class (12) was considered, e.g., in [8, 11]. Note that [8] is the first paper where the exhaustive study of admissible transformations of a class of differential equations was carried out. The equivalence group of class (12) is finite dimensional and consists of the transformations

$$\begin{aligned} \tilde{t} &= \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde{x} = \frac{\kappa x + \mu_1 t + \mu_0}{\gamma t + \delta}, \quad \tilde{u} = \frac{\kappa(\gamma t + \delta)u - \kappa\gamma x + \mu_1\delta - \mu_0\gamma}{\alpha\delta - \beta\gamma}, \\ \tilde{f} &= \frac{\kappa^2}{\alpha\delta - \beta\gamma} f, \end{aligned} \quad (13)$$

where the constant tuple $(\alpha, \beta, \gamma, \delta, \kappa, \mu_0, \mu_1)$ is defined up to a nonzero multiplier and satisfies the constraints $\alpha\delta - \beta\gamma \neq 0$ and $\kappa \neq 0$. The form of these transformations can be calculated directly or by means of the substitutions $F = f$, $\tilde{F} = \tilde{f}$, $H^1 = u$, $\tilde{H}^1 = \tilde{u}$, and $H^0 = \tilde{H}^0 = 0$ into (4). Since all transformations between any two fixed similar equations from (12) are exhausted by (13), class (12) is normalized.

The class of equations of the form

$$u_t + uu_x + (f(t, x)u_x)_x = 0 \quad (14)$$

with f running through the set of nonvanishing smooth functions of (t, x) admits the transformations

$$\begin{aligned} \tilde{t} &= T(t), \quad \tilde{x} = \varkappa\sqrt{|T_t|}x + X^0(t), \\ \tilde{u} &= \varkappa\frac{\sqrt{|T_t|}}{T_t}u + \varkappa\frac{T_{tt}\sqrt{|T_t|}}{2T_t^2}x + \frac{X^0}{T_t}, \quad \tilde{f} = \varkappa^2 f, \end{aligned} \quad (15)$$

where \varkappa is an arbitrary nonzero constant and the smooth functions T and X^0 of t satisfy the equation

$$\varkappa\sqrt{|T_t|}T_{tt}f_x + 2T_tX_{tt} - 2T_{tt}X_t = 0. \quad (16)$$

Unlike the previous classes, class (14) is not normalized. At the same time, its subclass singled out by the inequality $f_{xxx} \neq 0$ is normalized. In this case equation (16) split with respect to f_x leads to the constraints $X_{tx} = 0$ and $T_{tt} = 0$. Hence the associated equivalence groupoid is determined by the transformations

$$\tilde{t} = c_1^2 t + c_0, \quad \tilde{x} = \varkappa c_1 x + c_2 t + c_3, \quad \tilde{u} = \frac{\varkappa c_1 u + c_2 t + c_3}{c_1^2}, \quad \tilde{f} = \varkappa^2 f,$$

where c_0, c_1, c_2, c_3 , and \varkappa are arbitrary constants with $\varkappa c_1 \neq 0$, which form the equivalence group of this subclass.

The complementary subclass, which is defined by the constraint $f_{xxx} = 0$, i.e., $f = f^2(t)x^2 + f^1(t)x + f^0(t)$, possesses a wider equivalence groupoid. Namely, all admissible transformations in this subclass are of form (15), where the parameter-functions $T = T(t)$ and $X^0 = X^0(t)$ additionally satisfy the system of ODEs

$$4T_t T_{tt} f^2 + 2T_t T_{ttt} - 3T_{tt}^2 = 0,$$

$$\frac{\varkappa}{2} \sqrt{|T_t|} T_{tt} f^1 + T_t X_{tt}^0 - T_{tt} X_t^0 = 0,$$

and \varkappa is an arbitrary nonzero constant. Although the general solution of this system is parameterized by the arbitrary elements f^1 and f^2 in a nonlocal way,

$$T = \pm \int \left(C_2 \int e^{-2 \int f^2 dt} dt + C_1 \right)^{-2} dt + C_0,$$

$$X^0 = -\frac{\varkappa}{2} \int T_t \int \frac{\sqrt{|T_t|} T_{tt}}{T_t^2} f^1 dt dt + C_3 T + C_4,$$

the solution structure is the same for all values of the parameters. In other words, the subclass singled out from class (14) by the constraint $f_{xxx} = 0$ possesses a nontrivial generalized extended equivalence group, and it is normalized with respect to this group. See, e.g., [6, 16, 17, 20, 21, 22] for the related definitions and other examples of generalized extended equivalence groups.

Note that the class of equations $u_t + uu_x + f(t)u_{xx} = 0$, which differs from classes (12) and (14) only in arguments of f and is the intersection of these classes, is normalized with respect to the equivalence group (13) of the whole class (12). The group analysis of this class was performed in [3, 23].

5 Classical Burgers equation

To conclude, consider the class consisting of the single equation (1). It is well known [5, 2] that its linear counterpart is the heat equation $v_t + v_{xx} = 0$. The maximal Lie invariance algebra of the classical Burgers equation (1) is spanned by the vector fields [7]

$$\partial_t, \quad 2t\partial_t + x\partial_x - u\partial_u, \quad t^2\partial_t + tx\partial_x + (x - ut)\partial_u, \quad \partial_x, \quad t\partial_x + \partial_u.$$

The complete point symmetry group of equation (1) consists of the transformations

$$\tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde{x} = \frac{\kappa x + \mu_1 t + \mu_0}{\gamma t + \delta}, \quad \tilde{u} = \frac{\kappa(\gamma t + \delta)u - \kappa\gamma x + \mu_1\delta - \mu_0\gamma}{\alpha\delta - \beta\gamma},$$

where $(\alpha, \beta, \gamma, \delta, \kappa, \mu_0, \mu_1)$ is an arbitrary set of constants defined up to a nonzero multiplier, and $\alpha\delta - \beta\gamma = \kappa^2 > 0$. Up to composition with continuous point symmetries, this group contains the single discrete symmetry $(t, x, u) \rightarrow (t, -x, -u)$.

6 Conclusion

This paper deals with a hierarchy of normalized classes of generalized Burgers equations. Due to the normalization property, the group classification for these classes can be carried out using the algebraic method. There are several examples of normalized classes the equivalence groups of which are finite dimensional, which is an unexpected result.

It is important to emphasize the following phenomenon in the relationship between the classes of linearizable generalized Burgers equations (6) and linear equations (5) as well as their subclasses via the Hopf–Cole transformation. In view of the superposition principle for solutions of linear equations, class (5) possesses the wider set of admissible transformations than class (6). Transformations associated with the linear superposition depend on arbitrary elements of the corresponding initial equations. This obstacle destroys the normalization property of class (5), though this class is still semi-normalized in the usual sense. At the same time, the linear superposition principle has no counterpart for the linearizable equations among local transformations. This is why class (6) is normalized.

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